Student Workbook

Exploring Ellipses

Name: ______________________ Date: ____________ Class: ________

ESSENTIAL QUESTION

How does the relationship between a, b, and c define the shape and orientation of an ellipse, and how do we use the standard form equation to extract all key features?

Lesson Overview

An ellipse is the set of all points in a plane where the sum of distances to two fixed points (foci) is constant, equal to 2a. Every ellipse has a center, a major axis (the long way across) and a minor axis (the short way across), and the relationship $c^2 = a^2 - b^2$ connects the three defining lengths.

$$x^2/a^2 + y^2/b^2 = 1 \text{ (horizontal)}$$
$$x^2/b^2 + y^2/a^2 = 1 \text{ (vertical)}$$
$$c^2 = a^2 - b^2$$

Standard Form	Horizontal major axis (larger denom. under x^2): $x^2/a^2+y^2/b^2=1$, $a>b>0$. Vertical major axis (larger denom. under y^2): $x^2/b^2+y^2/a^2=1$. Centered at (h,k): replace x with (x−h), y with (y−k).
Vertices	Endpoints of the major axis, distance a from the center — always ON the ellipse.
Co-vertices	Endpoints of the minor axis, distance b from the center — always ON the ellipse.
Foci	Two fixed points INSIDE the ellipse, distance c from the center; $c^2 = a^2 - b^2$ (note the minus sign — not the Pythagorean plus).
Eccentricity	$e = c/a$, always $0 < e < 1$ for an ellipse. Near 0 = nearly circular; near 1 = very elongated.

Worked Examples

EXAMPLE 1

Identify all features of $x^2/25 + y^2/9 = 1$.

- $a^2 = 25 \rightarrow a = 5$; $b^2 = 9 \rightarrow b = 3$
- $c^2 = 25 - 9 = 16 \rightarrow c = 4$
- Horizontal major axis (larger denominator under x^2)

Answer: Center (0,0), vertices (±5,0), co-vertices (0,±3), foci (±4,0), e = 4/5

EXAMPLE 2

Write the equation of an ellipse with vertices at (±6, 0) and co-vertices at (0, ±4).

- $a = 6 \rightarrow a^2 = 36$; $b = 4 \rightarrow b^2 = 16$
- Vertices on the x-axis → horizontal major axis

Answer: $x^2/36 + y^2/16 = 1$

EXAMPLE 3

Find all features of $(x-2)^2/16 + (y+3)^2/4 = 1$.

- Center: $(2,-3)$; $a^2 = 16 \rightarrow a = 4$; $b^2 = 4 \rightarrow b = 2$
- $c^2 = 16 - 4 = 12 \rightarrow c = 2\sqrt{3}$; horizontal major axis
- Vertices: $(2\pm4,-3) = (6,-3), (-2,-3)$. Co-vertices: $(2,-3\pm2) = (2,-1), (2,-5)$

Answer: Center $(2,-3)$, vertices $(6,-3)$ & $(-2,-3)$, co-vertices $(2,-1)$ & $(2,-5)$, foci $(2\pm2\sqrt{3},-3)$

EXAMPLE 4

Write the equation of an ellipse with foci at $(0, \pm3)$ and vertices at $(0, \pm5)$.

- Vertical major axis (foci and vertices on the y-axis)
- $a = 5$, $c = 3 \rightarrow b^2 = a^2 - c^2 = 25 - 9 = 16$

Answer: $x^2/16 + y^2/25 = 1$

EXAMPLE 5

Convert $4x^2 + 9y^2 - 16x + 18y - 11 = 0$ to standard form.

- Group: $(4x^2 - 16x) + (9y^2 + 18y) = 11$
- Factor: $4(x^2 - 4x) + 9(y^2 + 2y) = 11$
- Complete the square: $4(x-2)^2 - 16 + 9(y+1)^2 - 9 = 11 \rightarrow 4(x-2)^2 + 9(y+1)^2 = 36$
- Divide by 36

Answer: $(x-2)^2/9 + (y+1)^2/4 = 1$; center $(2,-1)$, $a=3$, $b=2$

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Identify all features of $x^2/49 + y^2/36 = 1$.

Hint: Which denominator is larger? That determines the major axis direction. Then find $c^2 = a^2 - b^2$.

2 Write the equation of an ellipse with vertices at $(0, \pm7)$ and foci at $(0, \pm\sqrt{33})$.

Hint: Vertical major axis. Use $b^2 = a^2 - c^2$ to find b^2.

3 Find the center, vertices, and foci of $(x+1)^2/25 + (y-4)^2/9 = 1$.

Hint: Center is $(h, k) = (-1, 4)$. Then apply the standard formulas with $a^2 = 25$, $b^2 = 9$.

4 An ellipse has center (0,0), a vertex at (4, 0), and passes through the point $(2, 3\sqrt{3}/2)$. Find the equation.

Hint: Use a = 4, so $x^2/16 + y^2/b^2 = 1$. Substitute the point to solve for b^2.

5 Convert $9x^2 + 4y^2 - 54x + 8y + 49 = 0$ to standard form.

Hint: Group x and y terms, factor out coefficients, complete the square for each variable.

Key Vocabulary

Ellipse	Set of all points where the sum of distances to two fixed points (foci) equals 2a.
Major axis	The longer axis of the ellipse; length 2a.
Vertices	Endpoints of the major axis, distance a from center.
Co-vertices	Endpoints of the minor axis, distance b from center.
Foci (singular: focus)	Two fixed points inside the ellipse; $c^2 = a^2 - b^2$.
Eccentricity (e)	e = c/a; measures how 'stretched' the ellipse is $(0 < e < 1)$.

Quick Check Quiz

Circle the letter of the correct answer for each question.

1 **For $x^2/16 + y^2/9 = 1$, the foci are at:** Multiple Choice

- **A** $(\pm 4, 0)$
- **B** $(\pm 3, 0)$
- **C** $(\pm\sqrt{7}, 0)$
- **D** $(0, \pm\sqrt{7})$

2 **An ellipse has a = 5 and b = 3. Its eccentricity is:** Multiple Choice

- **A** 3/5
- **B** 4/5
- **C** 5/3
- **D** $\sqrt{34}/5$

3 **Which equation has a vertical major axis?** Multiple Choice

- **A** $x^2/9 + y^2/4 = 1$
- **B** $x^2/4 + y^2/9 = 1$
- **C** $x^2/9 + y^2/9 = 1$
- **D** $x^2/16 + y^2/9 = 1$

4 **For $(x-3)^2/25 + (y+2)^2/16 = 1$, the center is:** Multiple Choice

- **A** (3, 2)
- **B** (−3, 2)
- **C** (3, −2)
- **D** (−3, −2)

5 **The relationship between a, b, and c for an ellipse is:** Multiple Choice

- **A** $a^2 + b^2 = c^2$
- **B** $c^2 = a^2 - b^2$
- **C** $b^2 = a^2 + c^2$
- **D** $a^2 = b^2 - c^2$

Common Mistakes to Avoid

✗ Assuming the larger denominator always goes under x^2.	✓ The larger denominator determines the major axis direction. Under x^2 → horizontal; under y^2 → vertical.
✗ Using $c^2 = a^2 + b^2$ (like the Pythagorean theorem).	✓ For ellipses: $c^2 = a^2 - b^2$. The foci are INSIDE the ellipse, so $c < a$.
✗ Confusing vertices with foci.	✓ Vertices are the endpoints of the major axis (distance a). Foci are interior points (distance c). Always $c < a$.
✗ Forgetting to divide by the constant when completing the square.	✓ After completing the square, divide EVERY term by the constant on the right to get the standard form = 1.

Math Tips

★ Quick check: in standard form, the larger denominator tells you the major axis direction. Larger under x^2 → horizontal; larger under y^2 → vertical.

★ Memory trick for $c^2 = a^2 - b^2$: think of a right triangle inside the ellipse with hypotenuse a, leg b, and leg c.

★ Eccentricity $e = c/a$ is always between 0 and 1. Near 0 = nearly circular; near 1 = very elongated (like a comet orbit).

★ When completing the square, factor out the leading coefficient BEFORE completing the square: $4(x^2 - 4x + 4) = 4(x-2)^2$.

★ Always verify: $a > b > 0$ and $a > c > 0$. If these don't hold, recheck which value is a and which is b.